

TROPICAL ALGEBRAIC APPROACH TO CONSENSUS OVER NETWORKS

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ABSTRACT. In this paper we study the convergence of the max-consensus protocol. Tropical algebra is used to formulate the problem. Necessary and sufficient conditions for convergence of the max-consensus protocol over fixed as well as switching topology networks are given.

1. INTRODUCTION

The consensus problem addresses the question of whether it is possible to achieve a consensus among agents over a network while communicating only with the immediate neighbors. The consensus problem in multi-agent or distributed systems were addressed as early as in mid eighties (Tsitsiklis, 1984). The field has been dormant for a while, with sporadic appearances of seemingly unrelated publications (Reynolds, 1987; Vicsek *et al.*, 1995), until it saw a burst in the last few years (Jadbabaie *et al.*, 2003; Olfati-Saber & Murray, 2003; Olfati-Saber *et al.*, 2007; Ren *et al.*, 2007). The reason for a renewed interest in the field of consensus is due to the applications it finds in decentralized multi-agent coordination missions and the recent developments of concrete instances of such systems like wireless sensor networks and multiple unmanned aerial vehicles.

The average or the weighted average consensus of agents over networks, where the values of all nodes or agents in the network converges to a weighted average of their initial values, is what received wide attention of researches, and this and various extensions of it have been addressed extensively (Olfati-Saber *et al.*, 2007; Ren *et al.*, 2007). On the otherhand In a max-consensus, every agent in the network, communicating with only the immediate neighbors, should converge to a value which is maximum of all the initial values of the agents. Max-consensus has applications like decentralized leader selection and detection of faults in large networks. The problem of max-consensus over a network of fixed topology was addressed in (Olfati-Saber & Murray, 2003). However, as the max operation is not linear, its treatment in (Olfati-Saber & Murray, 2003) is different from that of linear protocols where convergence to consensus is proved through convergence of matrix products. We propose a different analysis of max-consensus protocol by forcing max operation to be linear through the use of tropical algebra (Baccelli *et al.*, 1992). By this we can give an analysis of the max-consensus protocol akin to the standard method of convergence analysis for

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linear consensus protocols, that is, analysis of products of adjacency matrices (Jadbabaie *et al.*, 2003). We also give necessary and sufficient conditions for convergence of max-consensus protocol under switching topology.

The rest of the paper is organized as follows. First, we state the max-consensus problem in its standard form. Then, we introduce the max-plus or the tropical algebra and thereafter reformulate the max-consensus problem. Under this new setting, we analyze the convergence of max-consensus protocol over a fixed network and give necessary and sufficient condition for this to occur. Then we study max-consensus under switching topologies. The necessary and sufficient condition for convergence is given for this case also.

2. BACKGROUND AND PRELIMINARIES

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph (digraph) with nodes $\mathcal{V} = \{1, \dots, N\}$ and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. For a node $i \in \mathcal{V}$, let \mathcal{N}_i denote the neighbor set of i . We have

$$\mathcal{N}_i = \{j : j \in \mathcal{V}, (j, i) \in \mathcal{E}\}.$$

By $(j, i) \in \mathcal{E}$, we mean an edge directed from node j to node i (node i receives information from node j). We consider graphs with self loops, that is, if $i \in \mathcal{V}$, then $(i, i) \in \mathcal{E}$. Let $x_i(k) \in \mathbb{R}$ be the value of node i at time step k . Then, the max-consensus protocol that we consider, for node i , is as follows:

$$x_i(k+1) = \max_{j \in \mathcal{N}_i} \{x_j(k)\}, \quad k \in \mathbb{Z}_{\geq 0}. \quad (2.1)$$

Now the max-consensus problem can be stated as follows:

Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, does there exist a $n \in \mathbb{N}$ such that for all $k \geq n$, $x_i(k) = \max_{j \in \mathcal{V}} \{x_j(0)\}$ for all $i \in \mathcal{V}$?

That is, the value at each node converges to a value which is equal to the maximum of all the initial nodal values.

The max-consensus problem is nonlinear under usual matrix algebra. Hence, we switch to an algebra to ‘linearize’ the max-consensus problem. This will enable us for natural extensions and generalizations like analysis of the max-consensus under switching topologies where the underlying graph changes with time. The suitable framework for the max-consensus problem is the tropical algebra. Here we give a brief overview of the tropical algebra (for details, see (Baccelli *et al.*, 1992; Izhakian, 2008)).

In tropical algebra, we consider a semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ called the tropical semiring with addition and multiplication defined as follows. For $a, b \in \mathbb{R} \cup \{-\infty\}$, define $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ and one can show that $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ has a semiring structure. $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called tropical semiring. The zero element of this semiring is $-\infty$ and the multiplicative identity is 0.

For the purpose of analyzing the max-consensus problem, we require only a tropical sub-semiring with two elements 0 and $-\infty$. We denote this binary tropical sub-semiring as $(\mathbb{T}, \oplus, \otimes)$, where $\mathbb{T} = \{0, -\infty\}$.

3. PROBLEM REFORMULATION

We reformulate the max consensus problem using tropical algebra. Towards this, we define the tropical adjacency matrix of a graph as follows.

Definition 3.1. *The tropical adjacency matrix of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a matrix $A \in \mathbb{T}^{N \times N}$, where $\mathcal{V} = \{1, \dots, N\}$, with $(i, j)^{\text{th}}$ entry defined as follows.*

$$(A)_{i,j} = \begin{cases} 0 & \text{if } i = j, \text{ or } (j, i) \in \mathcal{E} \\ -\infty & \text{otherwise.} \end{cases}$$

Since we consider graphs with self loops, all tropical adjacency matrices will have diagonal entries as zeros.

The tropical linear algebra operations involving matrices and vectors is as follows. For tropical adjacency matrices $A, B \in \mathbb{T}^{N \times N}$ and a vector $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, we have

$$\begin{aligned} (A \oplus B)_{ij} &= a_{ij} \oplus b_{ij} \\ &= \max\{a_{ij}, b_{ij}\} \end{aligned} \quad (3.2)$$

$$\begin{aligned} (A \otimes B)_{ij} &= \bigoplus_l (a_{il} \otimes b_{lj}) \\ &= \max_l \{a_{il} + b_{lj}\} \end{aligned} \quad (3.3)$$

$$\begin{aligned} (A \otimes \mathbf{x})_i &= \bigoplus_j (a_{ij} \otimes x_j) \\ &= \max_j \{a_{ij} + x_j\}, \end{aligned} \quad (3.4)$$

where $a_{ij} = (A)_{ij}$ and $b_{ij} = (B)_{ij}$ with $(\cdot)_{ij}$ denoting the $(i, j)^{\text{th}}$ element. The max-consensus protocol in Eq. (2.1) can now be rewritten using tropical algebraic operations as follows.

$$\begin{aligned} x_i(k+1) &= \max_{j \in \mathcal{N}_i} \{x_j(k)\} \\ &= \max \left\{ \max_{j \in \mathcal{N}_i} \{0 + x_j(k)\}, \max_{j \in \mathcal{V} \setminus \mathcal{N}_i} \{-\infty + x_j(k)\} \right\}. \end{aligned} \quad (3.5)$$

If A is the tropical adjacency matrix corresponding to graph \mathcal{G} , then from Definition 3.1 and Eq. (3.5), it follows that

$$\begin{aligned} x_i(k+1) &= \max_{j \in \mathcal{V}} \{(A)_{i,j} + x_j(k)\} \\ &= \bigoplus_{j=1}^n ((A)_{i,j} \otimes x_j(k)). \end{aligned}$$

Using the notation $\mathbf{x}(k) = (x_1(k), \dots, x_N(k))^T$, we have

$$x_i(k+1) = (A \otimes \mathbf{x}(k))_i, \quad \text{for all } i \in \mathcal{V}.$$

With the understanding that by product we mean tropical product, the max-consensus problem can be written as

$$\mathbf{x}(k+1) = A\mathbf{x}(k)$$

or equivalently

$$\mathbf{x}(k) = A^k \mathbf{x}(0), \quad (3.6)$$

where $\mathbf{x}(0) = (x_1(0), \dots, x_N(0))^T \in \mathbb{R}^n$ is the vector of initial values of nodes of \mathcal{G} and by A^k we mean $\underbrace{A \otimes \dots \otimes A}_{k \text{ times}}$.

Equation (3.6) is, in form, similar to the linear consensus protocol (Olfati-Saber *et al.*, 2007; Ren *et al.*, 2007) and its convergence can be analyzed by looking at the behavior of A^k for large k .

4. CONVERGENCE ANALYSIS

Towards deriving the convergence criterion for max-consensus protocol we first establish certain key properties of tropical adjacency matrix multiplication.

Definition 4.1. Given a tropical adjacency matrix $A \in \mathbb{T}^{N \times N}$, its dependency graph, denoted as $\mathcal{G}(A)$, is a graph $(\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \dots, N\}$ and $(i, j) \in \mathcal{E}$ if $(A)_{ji} = 0$.

In the sequel, we denote the edge set of the dependency graph $\mathcal{G}(A)$ as $\mathcal{E}(A)$. We have the following observations.

Lemma 4.2. Let $A, B \in \mathbb{T}^{N \times N}$ be tropical adjacency matrices and if any one of the following holds

- (1) $(j, i) \in \mathcal{E}(A)$,
- (2) $(j, i) \in \mathcal{E}(B)$, and
- (3) $\exists l$ such that $(l, i) \in \mathcal{E}(A)$ and $(j, l) \in \mathcal{E}(B)$

then $c_{ij} = 0$.

Proof. Case 1: We have $a_{ij} = 0$ as $(j, i) \in \mathcal{E}(A)$, and $b_{jj} = 0$ since it is diagonal element. This gives us that $a_{ij} + b_{jj} = 0$ and therefore from Eq. (4.8) (also using the fact that $c_{ij} \in \{0, -\infty\}$), we have

$$\begin{aligned} c_{ij} &= \max\left\{\max_{l \in \mathcal{V} \setminus j} (a_{il} + b_{lj}), (a_{ij} + b_{jj})\right\} \\ &= \max\left\{\max_{l \in \mathcal{V} \setminus j} (a_{il} + b_{lj}), 0\right\} = 0. \end{aligned}$$

Case 2: Follows from similar arguments as in Case 1.

Case 3: We have $a_{il} = 0$ as $(l, i) \in \mathcal{E}(A)$, and $b_{lj} = 0$ as $(j, l) \in \mathcal{E}(B)$. Therefore $a_{il} + b_{lj} = 0$ which leads us to the conclusion that $c_{ij} = 0$ using arguments similar to that in Case 1. \square

Now, we have the following lemma.

Lemma 4.3. Let $A, B \in \mathbb{T}^{N \times N}$ be tropical adjacency matrices. Then, $C = A \otimes B$ is also a tropical adjacency matrix. Moreover, $\mathcal{E}(C) \supseteq \mathcal{E}(A) \cup \mathcal{E}(B)$ and if there exists a node l such that $(l, i) \in \mathcal{E}(A)$ and $(j, l) \in \mathcal{E}(B)$, then $(j, i) \in \mathcal{E}(C)$.

Proof. Let a_{ij}, b_{ij} , and c_{ij} be the $(i, j)^{\text{th}}$ element of A, B , and C respectively. Since $C = A \otimes B$, from matrix multiplication rule, Eq. (3.3), we get

$$c_{ij} = \bigoplus_{l=1}^N (a_{il} \otimes b_{lj}) \quad (4.7)$$

$$= \max_l (a_{il} + b_{lj}). \quad (4.8)$$

Since $a_{ij}, b_{ij} \in \{0, -\infty\}$ and $(\{0, -\infty\}, \oplus, \otimes)$ is a semiring (that is, closed under operations \oplus and \otimes), from Eq. (4.7) we have $c_{ij} \in \{0, -\infty\}$. If $c_{ij} = 0$, then $(j, i) \in \mathcal{E}(C)$.

From Lemma 4.2 and the fact that A and B are tropical adjacency matrices, it is clear that $c_{ii} = 0$ for all $i \in \{1, \dots, N\}$. Thus C is a tropical adjacency matrix since all its entries belong to \mathbb{T} and all the diagonal entries are zeros. Also immediate are the other claims of the lemma. \square

From Lemma 4.3, we observe that the tropical adjacency matrix multiplication is non-commutative which is inherited from the non-commutativity of matrix multiplication. The essence of Lemma 4.3 is that the tropical adjacency matrix multiplication is superadditive with respect to the corresponding edge sets. Thus from this lemma, we have an immediate observation.

Proposition 4.4. *The 0 entries in the tropical adjacency matrices are not altered by adjacency matrix multiplications. Moreover, if A is any adjacency matrix $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$.*

In the above proposition and sequel, $\mathbf{0}$ corresponds to tropical adjacency matrix with all entries as zeros. An implication of above proposition is that, once a consensus is achieved, it becomes independent of the underlying graph structure. Thus, using a max-consensus protocol, a consensus, if achieved, is achieved in finite time. This is in contrast to other consensus protocols (average consensus protocol, for example) where the consensus is achieved asymptotically.

We have the following definitions.

Definition 4.5. *A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is strongly connected if for every $i, j \in \mathcal{V}$ there exists a sequence of nodes (called a path from i to j) in \mathcal{V} , $i = i_0, i_1, \dots, i_s = j$, such that $(i_{l-1}, i_l) \in \mathcal{E}$ for $l \in \{1, \dots, s\}$ and $s \leq N - 1$.*

Equivalently, in a strongly connected graph, it is possible to start at any node and reach any other node by following the directed edges of the graph.

Definition 4.6. *A completely connected graph is a graph in which there is a directed edge from every node to every other node.*

The tropical adjacency matrix that corresponds to a completely connected graph is the zero matrix $\mathbf{0}$. The shortest distance/path length from i to j is the least number of directed edges to follow from node i to reach node j .

Definition 4.7. *Let d_{ij} denote the shortest path length from node i to node j in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $i, j \in \mathcal{V}$. Then, diameter of the graph, $d = \max_{i, j \in \mathcal{V}} d_{ij}$.*

If the graph is not strongly connected, there may not be a path between some of the nodes and in that case, we assign a value of ∞ to the diameter.

Definition 4.8. For all $i, j \in \mathcal{V}$, node j is a p -neighbor of i if the shortest path length from node j to i is less than or equal to p . We denote the p -neighbor set of i as \mathcal{N}_i^p .

From the above definition, it follows that $\mathcal{N}_i^1 = \mathcal{N}_i$ and the following recursion formula holds

$$\mathcal{N}_i^p = \left(\bigcup_{l \in \mathcal{N}_i^{p-1}} \{j : j \in \mathcal{N}_l\} \right) \cup \mathcal{N}_i^{p-1}$$

We have the following proposition which immediately follows from the definitions of strongly connected graph, diameter of a graph, and the p -neighbor set of a node.

Proposition 4.9. For a strongly connected graph with vertex set \mathcal{V} and having diameter d , $\mathcal{N}_i^d = \mathcal{V}$ for all $i \in \mathcal{V}$.

For a strongly connected graph $\mathcal{G}(A)$ of diameter d , $\mathcal{G}(A^d)$ is a completely connected graph as shown in the proof of the following lemma, which leads to the main result.

Lemma 4.10. Let $A \in \mathbb{T}^{n \times n}$ be a tropical adjacency matrix. If $\mathcal{G}(A)$ is strongly connected, then $A^d = \mathbf{0}$ where d is the diameter of $\mathcal{G}(A)$

Proof. Let $\mathcal{G}_k = \mathcal{G}(A^k)$ for $k \in \mathbb{N}$. Let $\mathcal{N}_i^1 = \mathcal{N}_i$ be the neighbor set of node i in \mathcal{G}_1 . Using a special case of Lemma 4.3 (where B equal to A), it follows that the neighbor set of node i in \mathcal{G}_2 is \mathcal{N}_i^2 . Continuing the same argument, the neighbor set of node i in graph \mathcal{G}_k is \mathcal{N}_i^k . For, $k = d$, by Proposition 4.9, we have \mathcal{G}_d as a completely connected graph and thus the claim of lemma follows. \square

We have the following necessary and sufficient condition for convergence of max-consensus protocol.

Theorem 4.11. The max-consensus protocol in (2.1) converges for all initial conditions if and only if there exists a $k \in \mathbb{N}$ such that $A^k = \mathbf{0}$.

Proof. *if:*

If $A^k = \mathbf{0}$ for some $k \in \mathbb{N}$, then for all $i \in \mathcal{V}$

$$\begin{aligned} x_i(k+1) &= \left(A^k \mathbf{x}(0) \right)_i \\ &= \max\{0 + x_1(0), \dots, 0 + x_N(0)\} \\ &= \max_{j \in \mathcal{V}} \{x_j(0)\} \end{aligned}$$

only if:

Suppose A^k has at least one non-zero entry for all $k \in \mathbb{N}$. Let this be the

$(i, j)^{\text{th}}$ entry, that is, $(A)_{i,j} = -\infty$. So we have

$$\begin{aligned} x_i(k+1) &= \max\{0 + x_1(0), \dots, -\infty + x_j(0), \dots, 0 + x_N(0)\} \\ &= \max\{x_1(0), \dots, x_{j-1}(0), x_{j+1}(0), \dots, x_N(0)\} \end{aligned}$$

This means that the node i does not converge to the maximum of all initial nodal values if $x_j(0)$ happens to be the maximum. \square

The above result is used to prove the following theorem which is the main result of this section.

Theorem 4.12. *The max-consensus protocol in (2.1) converges for all initial conditions if and only if $\mathcal{G}(A)$ is strongly connected.*

Proof. if:

If $\mathcal{G}(A)$ is strongly connected, from Lemma 4.10, we have that $A^d = \mathbf{0}$ and thus using Theorem 4.11 it follows that a consensus is achieved in d time steps.

only if:

We use the fact that if $\mathcal{G}(A)$ is not strongly connected, then A is reducible (Fiedler, 2008; Brualdi & Ryser, 1991). That is, there exist a permutation σ of the numbering of nodes in \mathcal{V} such that the resultant adjacency matrix $\mathfrak{S}A\mathfrak{S}^T$ is of the form

$$\hat{A} = \begin{bmatrix} A_{11} & [-\infty] \\ A_{12} & A_{22} \end{bmatrix}$$

where \mathfrak{S} is the permutation matrix (Brualdi & Ryser, 1991) corresponding to σ , A_1 and A_2 are square matrices that are non-vacuous (dimension greater than or equal to 1), and $[-\infty]$ is a matrix of appropriate dimension with all entries as $-\infty$.

Claim 4.13. *There exist no $k \in \mathbb{N}$ such that, for a matrix \hat{A} of above form, $\hat{A}^k = \mathbf{0}$*

Proof. This follows from the tropical matrix multiplication of irreducible matrices (Brualdi & Ryser, 1991). In fact, such matrices retains above form after multiplication. \square

Thus if $\mathcal{G}(A)$ is not strongly connected, then there does not exist a $k \in \mathbb{N}$ such that $A^k = \mathbf{0}$ and thus a max-consensus is not achieved. \square

5. SWITCHING TOPOLOGY

Now, we look at the case of switching topology where the underlying graph changes during each time step. We consider a consensus protocol of the form

$$\mathbf{x}(k+1) = A_k \mathbf{x}(k) \tag{5.9}$$

where $A_k \in \mathbb{T}^{N \times N}$ is a tropical adjacency matrix for $k \in \mathbb{Z}_{\geq 0}$. Equivalently, we have

$$\mathbf{x}(k+1) = A_k A_{k-1} \cdots A_0 \mathbf{x}(0)$$

Theorem 5.1. *The consensus protocol in (5.9) converges for all initial conditions if and only if there exist a $k \in \mathbb{N}$ such that $A_k A_{k-1} \cdots A_0 = \mathbf{0}$.*

Proof. Proof is similar to that of Theorem 4.11 where A^k is replaced by $A_k A_{k-1} \cdots A_0$. \square

If A_k is drawn from a finite set $\{A_1, A_2, \dots, A_m\}$ of tropical adjacency matrices, an interesting question to ask is

Does there exist a finite sequence i_1, i_2, \dots, i_n with $1 \leq i_r \leq m$ and $r \in \{1, \dots, n\}$ such that the system in Eq. (5.9) converges in n time steps?

Before answering that we give the following useful definition.

Definition 5.2. *The graphs $\mathcal{G}_1(\mathcal{V}, \mathcal{E}_1), \dots, \mathcal{G}_m(\mathcal{V}, \mathcal{E}_m)$ with adjacency matrices A_1, \dots, A_m are called jointly strongly connected if the union graph $\mathcal{G}(A_1 \oplus \dots \oplus A_m) = \bigcup_{r=1}^m \mathcal{G}_r$, with vertex and edge set as $(\mathcal{V}, \bigcup_{r=1}^m \mathcal{E}_r)$, is strongly connected.*

Proposition 5.3. *Let $A, B \in \mathbb{T}^{N \times N}$ be tropical adjacency matrices. Then the graph $\mathcal{G}(A \otimes B)$ is strongly connected if and only if $\mathcal{G}(A)$ and $\mathcal{G}(B)$ are jointly strongly connected.*

Proof. if:

Follows from the superadditivity property of the tropical adjacency matrix multiplication proved in Lemma 4.3.

only if:

Follows from Claim 4.2 in the proof of Lemma 4.3. The claim is actually ‘if and only if’ although the ‘only if’ part is not required for Lemma 4.3. In fact, the three ways in which c_{ij} can become zero (refer to Claim 4.2) are the only ways in which it will be zero. As far as strong connectivity is concerned, condition 3 in Claim 4.2 is immaterial because reachability of a node is only what matters for strong connectivity in which case the third condition is just redundant. Thus, the strong connectivity of $\mathcal{G}(A \otimes B)$ is same as that of $\mathcal{G}(A) \cup \mathcal{G}(B)$. \square

Thus, if the graphs corresponding to the tropical adjacency matrices $\{A_1, \dots, A_m\}$ are jointly strongly connected, then $(A_1 \otimes \dots \otimes A_m)^n = \mathbf{0}$ for some $n \in \mathbb{N}$. Following theorem is the answer to the question of convergence of max-consensus protocol under switching topology (Eq. (5.9)).

Theorem 5.4. *Given a finite set of tropical adjacency matrices $\{A_1, \dots, A_m\} \subset \mathbb{T}^{N \times N}$, there exists a finite sequence A_{i_1}, \dots, A_{i_n} with $1 \leq i_r \leq m$ for all $r = \{1, \dots, n\}$ such that the system in Eq. (5.9) with $A_k \in \{A_1, \dots, A_m\}$ attains consensus for all initial conditions if and only if $\mathcal{G}(A_1), \dots, \mathcal{G}(A_m)$ are jointly strongly connected.*

Proof. Follows by extending the result of Proposition 5.3 to finite case and proceeding in the similar lines of proof of Theorem 4.12. \square

The above theorem has a couple of interesting corollaries. We give the following definition towards this.

Definition 5.5. *Given a finite set of matrices $\{A_1, \dots, A_m\}$, the matrix mortality problem asks the following question: Does there exist a finite sequence i_1, \dots, i_n with $1 \leq i_r \leq m$ and $r \in \{1, \dots, n\}$ such that $A_{i_1}, \dots, A_{i_n} = \mathbf{0}$?*

The matrix mortality problem is of great interest to the theoretical computer science community (Sipser, 2006). The matrix mortality problem is usually undecidable, that is, there does not exist an algorithm which can find such a sequence given a matrix set as input.

Corollary 5.6. *If the finite set of input matrices $\{A_1, \dots, A_m\}$ are tropical adjacency matrices, then the matrix mortality problem is decidable. Moreover, this can be done in polynomial time.*

From Theorem 5.4, we know that the matrix mortality problem has a positive answer if and only if A_1, \dots, A_m are jointly strongly connected. In that case, the zero matrix is obtained as $(A_1 \otimes \dots \otimes A_m)^n$ for some finite $n \in \mathbb{N}$. Another interesting corollary is as follows.

Corollary 5.7. *The semigroup (under binary operation \otimes which is associative) generated by the tropical adjacency matrices $\{A_1, \dots, A_m\} \subset \mathbb{T}^{N \times N}$ contains $\mathbf{0}$ if and only if $\bigcup_{r=1}^m \mathcal{G}(A_r)$ is strongly connected.*

The following remarks are in place.

A much more stronger result on the convergence of the max-consensus protocol can be achieved if the condition in our theorems ‘achieves consensus for all initial condition’ is relaxed. In particular, a system $\mathbf{x}(k+1) = A_k \mathbf{x}(0)$ achieves consensus for a particular input $\mathbf{x}(0)$ at time step $k+1$ if and only if there exists a vector $\mathbf{y} \in \mathbb{T}^N$ such that $A_k \mathbf{y} = \mathbf{0}$, where

$$y_i = \begin{cases} 0 & \text{if } x_i(0) = \max_j x_j(0) \\ -\infty & \text{otherwise} \end{cases}$$

In other words, if node i has a maximum initial value, the necessary and sufficient condition for a max-consensus to occur at the $(k+1)^{\text{th}}$ time step is that all elements in the i^{th} column of A^k are zeros or equivalently, the graph $\mathcal{G}(A^k)$ has a directed spanning tree rooted at node i .

In the case of the max-consensus protocol, if a consensus occurs, it will happen in a finite number of steps. This should be contrasted with the asymptotic convergence of weighted average consensus protocols (Olfati-Saber *et al.*, 2007; Ren *et al.*, 2007). This property enables us to give a stronger convergence criterion. In fact, we gave the necessary and sufficient condition for convergence under switching topology. However, this is difficult in general for the weighted average consensus case as one has to consider infinite product of matrices (as opposed to product of finite number of matrices in the max-consensus case) while analyzing convergence. Thus, in case of weighted average consensus, one is forced to impose stronger requirements on the matrices to achieve convergence of the infinite matrix products. This will, in general, result in obtaining only a sufficiency condition for convergence.

A max-consensus, scheduled at regular intervals, over large networks can be used to detect network faults. This is practical as most of the

naturally occurring networks like random networks (Erdős & Rényi, 1960), small world networks (Watts & Strogatz, 1998), and scale free networks (Barabási & Albert, 1999) usually have very small diameters (Newman *et al.*, 2006). Since the number of time steps for max-consensus algorithm to converge is equal to the diameter of the underlying network, consensus is achieved fast in these networks due to the ‘small diameter property’. A failure to attain consensus in the specified time is a indication to the presence of a network fault.

The matrix operations in the semiring $(\mathbb{T}, \max, +)$ is exactly same as the matrix operations in the boolean semiring $(\{0, 1\}, \text{OR}, \text{AND})$ (Hammer, 1968) as there is a ring-isomorphism between these two semirings $((\mathbb{T}, \max, +)$ is isomorphic to $(\{0, 1\}, \text{OR}, \text{AND})$). Thus, although the max-consensus problem cannot be posed using boolean semiring, all the matrix multiplication properties used to prove convergence can be identically obtained by working in boolean semiring (See (Hammer, 1968) for properties of boolean adjacency matrix multiplication).

6. CONCLUDING REMARKS

We analyzed the convergence of max consensus protocol with both fixed and switching topologies. The observation that tropical algebra gives a natural way to formulate this problem enabled an analysis of the convergence in terms of tropical matrix products which could be easily extended to switching topology case. We gave the necessary and sufficient condition for max-consensus to occur in fixed as well as switched topology networks.

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